NEW METHODS IN THE THEORY OF RADIATIVE TRANSPORT.
II. ANISOTROPIC SCATTERING

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The propagation of radiation in an anisotropically scattering layer is considered using the method developed in the first paper of the current series [1].

The solution of problems involving the propagation of radiation in an anisotropically scattering medium usually relies on expansions of the intensity and scattering indicatrix of the radiation by an elementary volume of the medium in series of Legendre polynomials [2, 3]. The expansion coefficients for the intensity of diffusely reflected radiation are given by functions of the Ambartsumyan type, defined by a system of integral equations. The complexity of this solution means that one must resort to studying the simplest cases of anisotropic scattering. For example, in the monograph [2], the general solution of the radiative transport equation in a semiinfinite medium was obtained for a linear anisotropy of the scattering indicatrix

$$p(\gamma) = 1 + x\cos\gamma, \tag{1}$$

and numerical results were given only for x = 1.

The numerical solution of the radiative transport equation with a nonspherical scattering indicatrix involves a large amount of execution time [4-7]. Therefore, recent interest has been shown in the development of approximate ways of taking into account the effect of anisotropic scattering [6-8]. For example, a method of treating anisotropic scattering was discussed in [8] for the transport approximation to a radiating two-phase medium.

When the radiation propagates in a medium with a nonuniform distribution of sources, the method discussed in [1] is used.

Then the solution for the case of an infinite medium reduces to determining the multiple scattering function

$$f(\tau_0, \mu) = \beta\int_0^\infty G(t, -\mu)e^{-t/\mu}\frac{dt}{\mu} + \frac{l}{2k}\left(\frac{1}{1-k\mu} - \frac{R}{1+k\mu}\right)\int_0^\infty h(t)e^{-kt}dt - \frac{l\mu^2}{1-k^2\mu^2}\int_0^\infty h(t)e^{-t/\mu}\frac{dt}{\mu}, \tag{2}$$

where

$$G(t, \mu) = \frac{\lambda^2}{2}\int_{-1}^{1} p(\mu, \mu')I^{(1)}(\tau, \mu')d\mu'; \tag{3}$$

$$h(t) = 2[g_1(t) + g_2(t)] - \frac{d}{dt}[g_1(t) - g_2(t)];$$

$$g_1(t) = \beta\int_0^1 G(t, +\mu)d\mu,\ g_2(t) = \beta\int_0^1 G(t, -\mu)d\mu,$$

$$k^2 = 4(1-l),\ l = \beta a\lambda,\ \beta = \frac{1}{1-\lambda(1-a)},\ R = \frac{2-k}{2+k}. \tag{4}$$

From the quantity $f(\tau_0, \mu)$ we can easily find the intensity of diffusely reflected radiation:

$$I(0, -\mu) = \lambda(1+\Delta)I^{(1)}(0, -\mu),\ \Delta = f(0, -\mu)/\lambda I^{(1)}(0, -\mu). \tag{5}$$

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TABLE 1. Values of the Function $R(\mu, \mu)$

μ	$\lambda=0,5$			$\lambda=0,7$			$\lambda=0,9$		
	(7)	[2]	(2)	(7)	[2]	(2)	(7)	[2]	(2)
0,2	1,15	1,21	1,19	1,30	1,35	1,34	1,54	1,63	1,60
0,4	1,15	1,21	1,18	1,39	1,47	1,44	1,89	2,03	1,98
0,6	0,98	1,05	1,05	1,32	1,43	1,40	2,09	2,25	2,21
0,8	0,77	0,87	0,81	1,16	1,27	1,23	2,16	2,43	2,31
1,0	0,44	0,49	0,45	0,89	0,99	0,92	2,44	2,44	2,27

It is known that [2]

$$I^{(1)}(0, -\mu) = \frac{p(-\mu, \mu_0)}{4(\mu+\mu_0)}\mu_0 I_0. \tag{6}$$

We note that the use of a correction for multiple scattering in the isotropic approximation [1] leads to a positive result for a scattering indicatrix of the type (1). We compare, in Table 1, the exact and approximate results for the coefficient of diffuse reflection $\rho(\mu, \mu)$ with the coefficient $8\mu/\lambda$:

$$R(\mu, \mu) = \frac{8\mu}{\lambda}\rho(\mu, \mu) \cong p(-\mu, \mu) + \Delta^{is}(\mu). \tag{7}$$

According to [1], the function $G(t, \mu)$ in (2) can be written in the following form for a scattering indicatrix of the type (1):

$$G(t, \mu) = G_0(t) + x(\mu+\mu_0)G_1(t) + x^2\mu\mu_0 G_2(t). \tag{8}$$

Here

$$G_n(t) = \frac{\lambda^2}{8}\mu_0 I_0\left\{\int_0^1 \frac{e^{-\beta t/\mu_0} - e^{-\beta t/\mu'}}{\mu_0-\mu'}(\mu')^n d\mu' + (-1)^n e^{-\beta t/\mu_0}\int_0^1 \frac{(\mu')^n d\mu'}{\mu_0+\mu'}\right\} \quad (n = 0, 1, 2). \tag{9}$$

Then

$$h(t) = 4\beta[G_0(t) + x\mu_0 G_1(t) + \beta x G_3(t)], \tag{10}$$

$$G_3(t) = \frac{\lambda^2}{32} I_0\left\{\int_0^1 \frac{\mu' e^{-\beta t/\mu_0} - \mu_0 e^{-\beta t/\mu'}}{\mu_0-\mu'}(1+x\mu_0\mu')\,d\mu' - e^{-\beta t/\mu_0}\int_0^1 \frac{1-x\mu_0\mu'}{\mu_0+\mu'}\mu' d\mu'\right\}. \tag{9a}$$

The relations (2), (9), and (9a) show that the determination of the function $f(\tau_0, \mu)$ for the case of anisotropic scattering reduces to the evaluation of the integrals:

$$J_m(z) = z\int_0^\infty G_m(t)e^{-zt}dt \quad (m = 0, 1, 2, 3). \tag{11}$$

Indeed,

$$\int_0^\infty G(t, -\mu)e^{-t/\mu}\frac{dt}{\mu} = J_0\left(\frac{1}{\mu}\right) + x(\mu_0-\mu)J_1\left(\frac{1}{\mu}\right) - x^2\mu_0\mu J_2\left(\frac{1}{\mu}\right),$$
$$\frac{1}{k}\int_0^\infty h(t)e^{-kt}dt = \frac{4\beta}{k^2}[J_0(k) + x\mu_0 J_1(k) + \beta x J_3(k)], \tag{12}$$
$$\int_0^\infty h(t)e^{-t/\mu}\frac{dt}{\mu} = 4\beta\left[J_0\left(\frac{1}{\mu}\right) + x\mu_0 J_1\left(\frac{1}{\mu}\right) + \beta x J_3\left(\frac{1}{\mu}\right)\right].$$

We evaluate the integrals (11):

$$J_0(z) = \frac{\lambda^2}{8}\mu_0 J_0\frac{1}{\beta+z\mu_0}\left(\beta\ln\frac{\beta+z}{\beta} + z\mu_0\ln\frac{1+\mu_0}{\mu_0}\right), \tag{13}$$

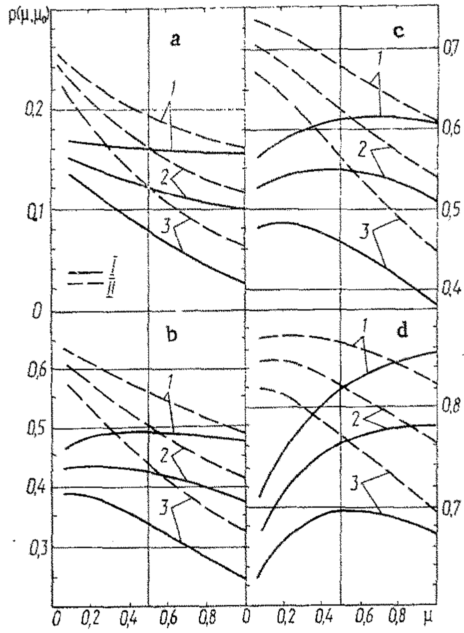

Fig. 1. Angular distribution of the coefficient of diffuse reflection: a) $\lambda = 0.5$; b) $\lambda = 0.9$; c) $\lambda = 0.95$; d) $\lambda = 0.99$; I: $\mu_0 = 1.0$; II: $\mu_0 = 0.6$; 1) $x = -1$; 2) $x = 0$; 3) $x = +1$.

TABLE 2. Dependence of the Coefficient of Diffuse Reflection $\rho(\mu, \mu_0)$ on the Anisotropy Parameter x ($\mu_0 = \mu = 1.0$)

x	λ							
	0,4	0,5	0,6	0,7	0,8	0,9	0,95	0,99
—1	0,117	0,155	0,201	0,260	0,342	0,478	0,608	0,856
0	0,070	0,098	0,133	0,182	0,254	0,384	0,515	0,780
1	0,016	0,028	0,048	0,080	0,152	0,255	0,387	0,676

TABLE 3. Values of the Function $\Delta(\mu, \mu_0, x/\Delta(\mu, \mu_0, x = 0)$ for $\mu_0 = 0.9$

μ	x								
	—1	0,5	1	—1	0,5	1	—1	0,5	1
	λ=0,4			λ=0,9			λ=0,99		
0,1	1,02	0,97	0,93	0,93	1,04	1,09	0,89	1,07	1,16
0,2	0,89	1,05	1,09	0,83	1,11	1,24	0,80	1,14	1,31
0,4	0,71	1,18	1,41	0,71	1,23	1,58	0,69	1,26	1,67
0,6	0,60	1,33	1,92	0,62	1,38	2,15	0,61	1,41	2,30
0,8	0,51	1,50	3,02	0,55	1,56	3,44	0,54	1,60	3,71
1,0	0,49	1,72	8,01	0,50	1,80	9,39	0,49	1,85	10,2

$$J_1(z) = \frac{\lambda^2}{8}\mu_0 J_0 \frac{1}{\beta + z\mu_0}\left(\beta - z\mu_0 - \frac{\beta^2}{z}\ln\frac{\beta+z}{\beta} + z\mu_0^2\ln\frac{1+\mu_0}{\mu_0}\right), \tag{13}$$

$$J_2(z) = \frac{\lambda^2}{8}\mu_0 J_0 \frac{z}{\beta + z\mu_0}\left[\frac{\mu_0}{2}(1-2\mu_0) + \frac{\beta}{2z^2}(z-2\beta) + \frac{\beta^3}{z^3}\ln\frac{\beta+z}{\beta} + \mu_0^3\ln\frac{1+\mu_0}{\mu_0}\right],$$

$$J_3(z) = \frac{\lambda^2}{32} J_0 \frac{1}{\beta + z\mu_0}\left[x\mu_0^2(\beta - \mu_0 z) - 2\mu_0 z + \frac{\mu_0\beta}{z}(z - x\mu_0\beta)\ln\frac{\beta+z}{\beta} + z\mu_0^2(1 + x\mu_0^2)\ln\frac{1+\mu_0}{\mu_0}\right].$$

Calculation of the function $R(\mu, \mu)$ according to (2), (12), and (13) demonstrates the better accuracy of these relations in comparison with the approximate expression (7) (see Table 1). These relations can be used to study the coefficient of diffuse reflection $\rho(\mu, \mu_0)$ and the effect of the multiple scattering function for different values of the scattering anisotropy parameter x.

In [1] the case x = 0 (isotropic scattering) was considered in detail and the high accuracy (≤2%) of the method was demonstrated. For $-1 \leq x \leq 1$, and for small values of μ_0 and μ, the nature of the scattering indicatrix does not affect (to within an error of 0.5%) the magnitude of the coefficient of diffuse reflection $\rho(\mu, \mu_0)$. As μ_0 and μ increase, the quantity $\rho(\mu, \mu_0)$ begins to depend strongly on x (for $\lambda \lesssim 0.9$). For a strongly scattering medium, Fig. 1 shows that the dependence of the coefficient of diffuse reflection $\rho(\mu, \mu_0)$ on the angle of observation becomes quite different for the two cases $\mu_0 = 1$ and $\mu_0 = 0.6$. Table 2 shows the results of the calculations for $\rho(\mu, \mu_0)$ for the example $\mu_0 = \mu = 1.0$. In Fig. 1 we also show the angular distribution of the coefficient of reflection $\rho(\mu, \mu_0)$ for x = −1, 0, +1. The data show that it is necessary to take into account the anisotropy of the scattering in the calculations of $\rho(\mu, \mu_0)$. The degree of anisotropy even more strongly affects the multiple scattering function. This is confirmed by the data of Table 3 (the values of the quantity $\Delta(\mu, \mu_0, x = 0)$ are presented in [1].

In spite of the sharp difference between the values of Δ for different λ (e.g., when $\mu_0 = 0.9$, $\mu = 1.0$, and x = 1.0, we have $\Delta|_{\lambda=0.4} = 3.12$ and $\Delta|_{\lambda=0.99} = 51.07$), the nature of the variation of the function $\Delta(\mu, \mu_0, x)/\Delta(\mu, \mu_0, x = 0)$ on the angle of observation is nearly identical for all values of λ and strongly depends on the anisotropy parameter x.

NOTATION

$I(\tau, \mu)$, radiation intensity at the point τ and in the direction $\theta = \arccos \mu$; $I^{(1)}{}_{\tau,\mu}$, intensity of singly scattered radiation; I_0, intensity of radiation incident from outside at angle $\theta_0 = \arccos \mu_0$; $\lambda = \sigma/(\kappa + \sigma)$, probability of quantum survival; κ, σ, coefficients of absorption and scattering of the medium; $\tau = \int_0^z (\varkappa + \sigma) dz$, optical thickness of the layer; $p(\mu, \mu')$, scattering indicatrix of radiation by an elementary volume of the medium; a, double-hemisphere fraction of backward scattering; $f(\tau, \mu)$, multiple-scattering function; $\rho(\mu, \mu_0)$, coefficient of diffuse reflection.

LITERATURE CITED

1. K. S. Adzerikho, N. V. Podluzhnyak, and Yu. V. Khodyko, "New methods in the theory of radiative transport. I. General solution. Semiinfinite layer," Inzh.-Fiz. Zh., 41, No. 6, 1083-1093 (1981).
2. V. V. Sobolev, Transport of Radiation Energy in the Atmospheres of the Stars and Planets [in Russian], GITTL, Moscow (1956).
3. V. V. Sobolev, Light Scattering in Planetary Atmospheres [in Russian], Nauka, Moscow (1972).
4. J. Canosa and R. H. Rehafiel, "A direct solution of the radiative transfer equation: application to Rayleigh and Mie atmospheres," JQSRT, 13, No. 1, 21-39 (1973).
5. R. P. Hemenger, "Solution of modified transport equation for multiple scattering in suspensions of highly anisotropic scatterers," JOSA, 64, No. 4, 503-509 (1974).
6. Yu. A. Biryukov and Yu. V. Krylov, "On the modifications to the Sobolev approximation for the case of strong anisotropic scattering," Izv. Akad. Nauk, Fiz. Atmos. Okeana, 10, No. 11, 1231-1235 (1974).
7. A. Dayan and C. L. Tien, "Radiative transfer with anisotropic scattering in an isothermal slab," JQSRT, 16, No. 2, 1113-1125 (1976).
8. K. S. Adzerikho and V. P. Nekrasov, "On the effect of anisotropic scattering on the luminosity of a two-phase medium," Inzh.-Fiz. Zh., 34, No. 5, 894-896 (1978).